5.8 bifurcations

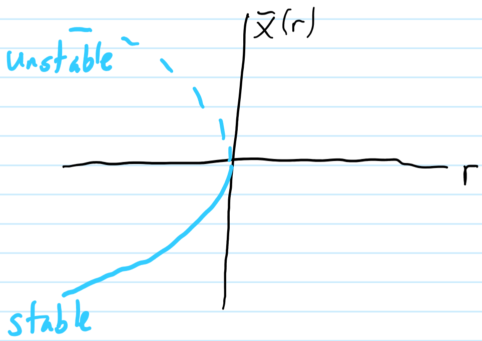
Monday, March 29, 2021 11:43 AM

Bifurcations in 1st-order differential equations

Let $\frac{dx}{dt} = f(x, r)$, where r is the bifurcation parameter and $\bar{x}(r)$ is the set of equilibria given r.

3 types of bifurcations (viz. 2.7)

I. saddle node (blue sky)



Ex. $\frac{dx}{dt} = r + x^2 = f(x,r)$ at $r = 0$

$0 = r + \bar{x}^2$

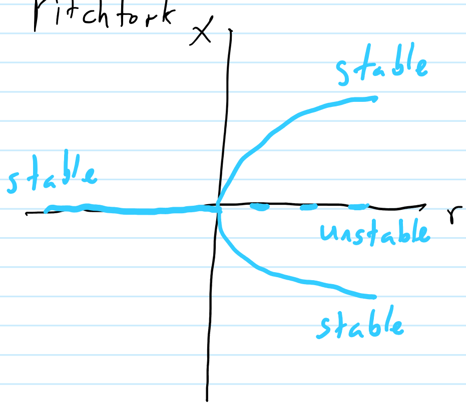
$\Rightarrow \bar{x} = \pm\sqrt{-r} \in \mathbb{R}$ when $r \le 0$, $\bar{x} = \emptyset$ when $r > 0$

Holding r constant, let $f(x) = r + x^2$

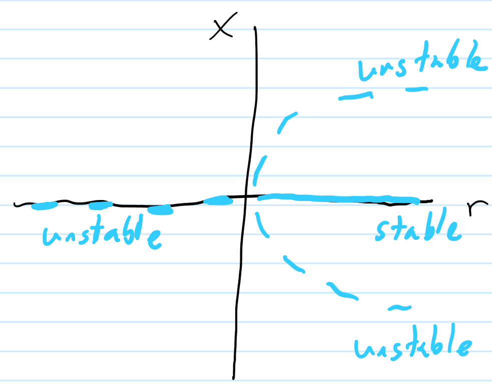
$f'(x) = 2x$

Case $r \le 0$ $\begin{cases} \Rightarrow f'(\sqrt{-r}) = 2\sqrt{-r} > 0, \text{ so unstable} \\ \Rightarrow f'(-\sqrt{-r}) = -2\sqrt{-r} < 0, \text{ so stable} \end{cases}$

II. Pitchfork



supercritical



subcritical

Ex. $\frac{dx}{dt} = rx - x^3 = f(x, r)$

$0 = r\bar{x} - \bar{x}^3$

$0 = \bar{x}(r - \bar{x}^2)$

Letting $f(x) = rx - x^3$ for r constant,

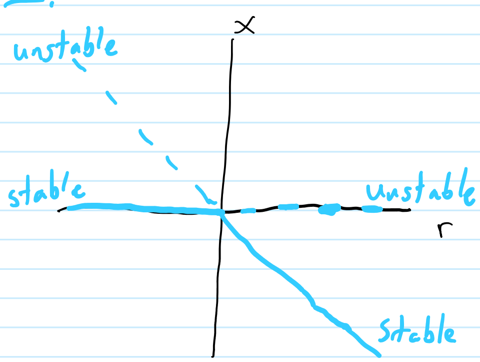
$f'(x) = r - 3x^2$

$f'(0) = r$

$$0 = r\bar{x} - \bar{x}^3$$
$$0 = \bar{x}(r - \bar{x}^2)$$
$$\Rightarrow \bar{x} = 0, \ \pm\sqrt{r}$$

$$f'(x) = r - 3x^2$$
$$f'(0) = r$$
$$f'(\sqrt{r}) = r - 3r = -2r, \quad r > 0$$
$$f'(-\sqrt{r}) = r - 3r = -2r, \quad r > 0$$

III. Transcritical



Ex. $\dfrac{dx}{dt} = rx + x^2$

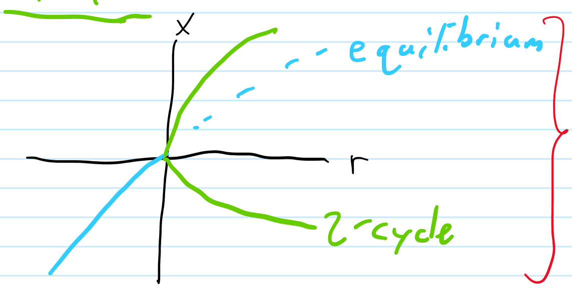
$$0 = r\bar{x} + \bar{x}^2$$
$$0 = \bar{x}(r + \bar{x})$$
$$\bar{x} = 0, -r$$

Let $f(x) = rx + x^2$

$$f'(x) = r + 2x$$
$$f'(0) = r$$
$$f'(-r) = -r$$

Systems of ODEs and the Hopf bifurcation

Recall: Period-doubling bifurcations for difference equations



But 1st-order ODEs can't be periodic, so inapplicable

Define: A **Hopf bifurcation** is a critical pt where stability changes and a periodic solution appears.

Ex. 5.19 (linear system)

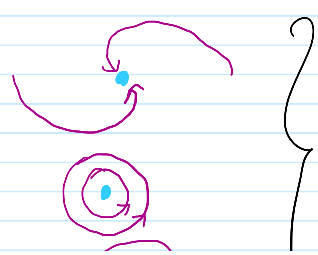
$$\frac{dx}{dt} = rx - y$$
$$\frac{dy}{dt} = x + ry$$

$$\dot{X}(t) = \begin{bmatrix} r & -1 \\ 1 & r \end{bmatrix} X(t), \quad X(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$$

$$J(0,0) = \begin{bmatrix} r & -1 \\ 1 & r \end{bmatrix} \qquad \lambda_{1,2} = r \pm i$$

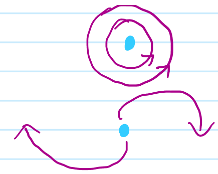
If $r < 0$, then $(0,0)$ is a stable spiral

If $r = 0$, then $(0,0)$ is a neutral centre



If $r = 0$, then $(0,0)$ is a neutral centre

If $r > 0$, then $(0,0)$ is an unstable spiral

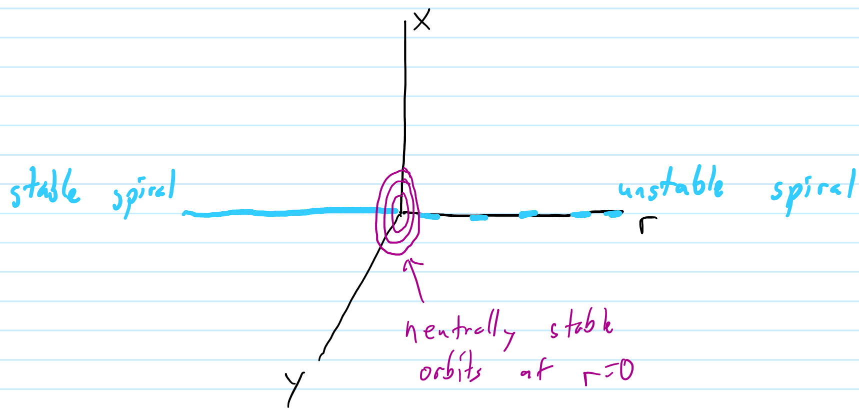


infinitely many neutrally stable closed orbits around equilibrium

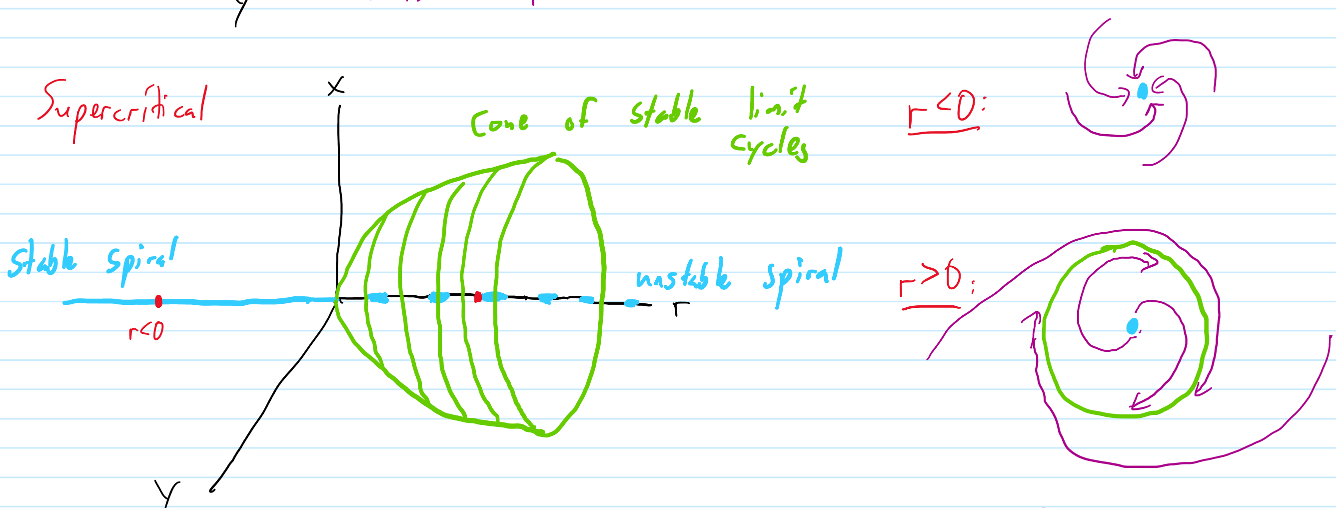
$\Rightarrow$ (degenerate) Hopf bifurcation at $r = 0$.

Classification of Hopf bifurcations

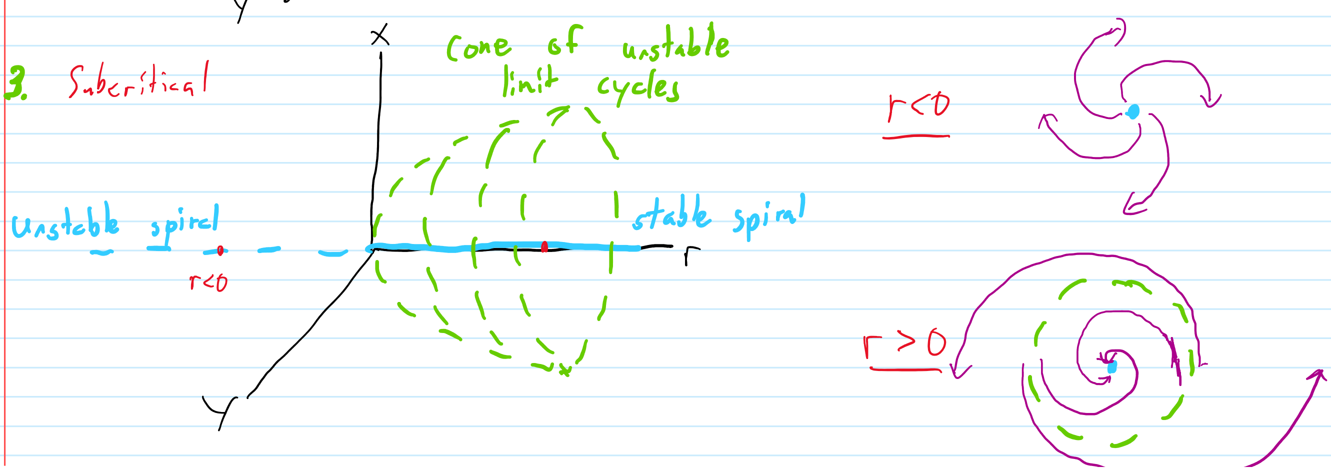
1. (degenerate) At bifurcation value, infinitely many neutrally stable closed orbits around equilibrium.

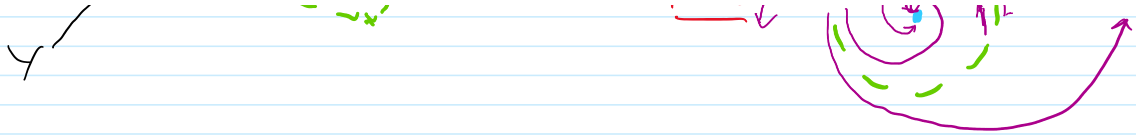


2. Supercritical



3. Subcritical





Thm 5.10 (Hopf bifurcation thm)

Consider the system $\frac{dx}{dt} = f(x,y,r)$, $\frac{dy}{dt} = g(x,y,r)$, or equiv. $\dot{X}(t) = \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \end{pmatrix} = \begin{pmatrix} f(x,y,r) \\ g(x,y,r) \end{pmatrix}$

WLOG, assume the origin $(0,0)$ is an equilibrium (otherwise, translate coordinates)

We can rewrite the system, separating out linear terms

$$\begin{aligned} \frac{dx}{dt} &= a_{11}(r)x + a_{12}(r)y + f_1(x,y,r) \\ \frac{dy}{dt} &= a_{21}(r)x + a_{22}(r)y + g_1(x,y,r) \end{aligned} \iff \dot{X}(t) = \begin{bmatrix} a_{11}(r) & a_{12}(r) \\ a_{21}(r) & a_{22}(r) \end{bmatrix} X(t) + \begin{bmatrix} f_1(x,y,r) \\ g_1(x,y,r) \end{bmatrix}$$

Assume $J(r) = \begin{bmatrix} a_{11}(r) & a_{12}(r) \\ a_{21}(r) & a_{22}(r) \end{bmatrix}$ is the Jacobian at $(0,0)$, valid for any small $|r|$.

Assume f_1, g_1 have continuous 3rd derivatives in x and y.

In addition, assume that $\alpha(r) \pm i\beta(r)$ are the eigenvalues of $J(r)$ for small $|r|$.

with $\alpha(0) = 0$ and $\beta(0) \neq 0$ such that

the eigenvalues cross the imaginary axis with nonzero speed

$$\frac{d\alpha}{dr}(0) \neq 0$$

Then, in any open set $U \subseteq \mathbb{R}^2$ s.t. $(0,0) \in U$ and for any $r_0 > 0$,

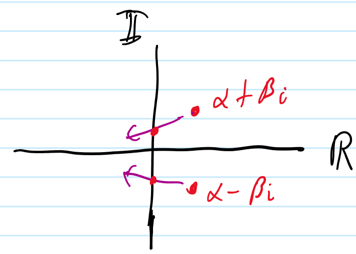
$\exists\ \bar{r}$ s.t. $|\bar{r}| < r_0$ s.t. there exists a periodic solution for $r = \bar{r}$ in U.

(with approximate period $T \approx \frac{2\pi}{\beta(0)}$)

Intuitively Consider the Jacobian $J(r)$ at the origin, where the origin is an equilibrium.

If, at $r=0$, eigenvalues are purely imaginary, but eigenvalues have differently signed nonzero real part for $r<0$ and $r>0$, then there is a Hopf bifurcation.

then there is a Hopf bifurcation.



Ex.

$$\frac{dx}{dt} = rx + y$$

$$\frac{dy}{dt} = -x + ry - y^3$$

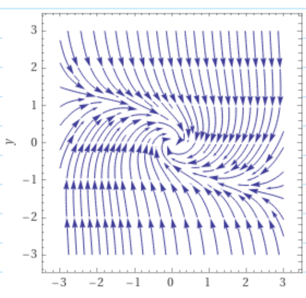
$$J(x,y,r) = \begin{bmatrix} r & 1 \\ -1 & r-3y^2 \end{bmatrix}$$

$$J(0,0,r) = \begin{bmatrix} r & 1 \\ -1 & r \end{bmatrix}$$

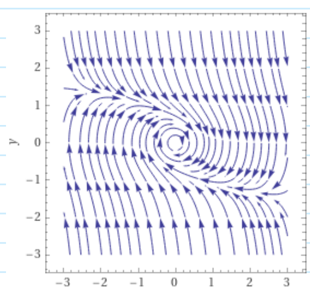
$$\lambda_{1,2} = r \pm i$$

By Hopf bifurcation theorem, there is a Hopf bifurcation at $r=0$.

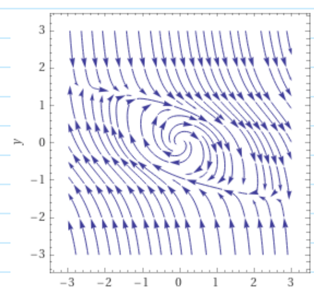
$r=-0.5$



$r=0$



$r=0.5$



WolframAlpha command: streamplot (rx+y, -x+ry-y^3) x=-2..2, y=-2..2, r=0.5